

Ghosts without runaway instabilities

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I. Introduction

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II. Ghostifying (integrable) models

III. Stable motion of a ghost interacting with a positive energy degree of freedom

IV. Conclusions



C.D., S. Mukohyama, A. Vikman,
PRL 128 (2022) 4, 041301.

C.D., A. Held, S. Mukohyama, A. Vikman
JCAP 11 (2023) 031.

C.D. A. Held, S. Mukohyama, A. Vikman,
in preparation.

I. Introduction

« Ghosts » are (usually considered as) degrees of freedom with « wrong sign » kinetic energies.



e.g. in a mechanical model, if a standard degree of freedom x has a free Hamiltonian

$$H = p_x^2$$

A ghost y would have the Hamiltonian

$$H = -p_y^2$$



In a scalar field theory, if a standard degree of freedom ϕ has a free Lagrangian

$$L = \partial_\mu \phi \partial^\mu \phi$$

A ghost ψ would have the Lagrangian

$$L = - \partial_\mu \psi \partial^\mu \psi$$



In field theories, ghosts being associated with negative energies, their existence is linked with violations of energy conditions

- Free ghosts are not problematic
- E.g. a stable positive energy x with Hamiltonian

$$H_x = p_x^2 + \omega_x^2 x^2$$

Can coexist stably with a ghost y with Hamiltonian

$$H_y = -p_y^2 - \omega_y^2 y^2$$

- Or similarly, e.g.

$$H = + p_x^2 - \omega_x^2 x^2 + \Omega_x^4 x^4 \\ - p_y^2 + \omega_y^2 y^2 - \Omega_y^4 y^4$$

- Free ghosts are not problematic
- The trouble arises when they interact with (standard) non ghost degrees of freedom
- Ghosts should at least interact gravitationally with standard degrees of freedom



Standard lore:

These interactions lead to instabilities which can already be seen @ classical level in the form of runaways solutions.

NB: Various type of instabilities (stabilities) can be considered

@ Classical level



Local stability :
the motion stays close to its initial values



Global stability :
No runaway solutions: the motion stays
bounded for all initial data

@ Quantum level



No unitarity, vacuum decay etc..

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Our paper

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This talk

NB:

For positive degrees of freedom, x and y , the boundedness of the motion can be obtained from the form of the Hamiltonian

e.g.
$$H = \underbrace{p_x^2 + p_y^2}_{\text{kinetic energy}} + V(x, y)$$

If V is bounded below, the conservation of H implies that p_x and p_y are bounded

NB:

For positive degrees of freedom, x and y , the boundedness of the motion can be obtained from the form of the Hamiltonian

e.g.
$$\text{If } H = \underbrace{p_x^2 + p_y^2 + V(x, y)}$$

If V is bounded below, the conservation of H implies that p_x and p_y are bounded



Does not work for a system with a ghost

$$H = p_x^2 - p_y^2 + V(x, y)$$

Yet, ghosts could have some interest to describe the real world



Higher derivative theory, some with potentially interesting applications (e.g. [Stelle 1976](#) for gravity, see also [Lee, Wick, 1970](#)) feature generically ghosts ([Ostrogradsky, 1850](#)).

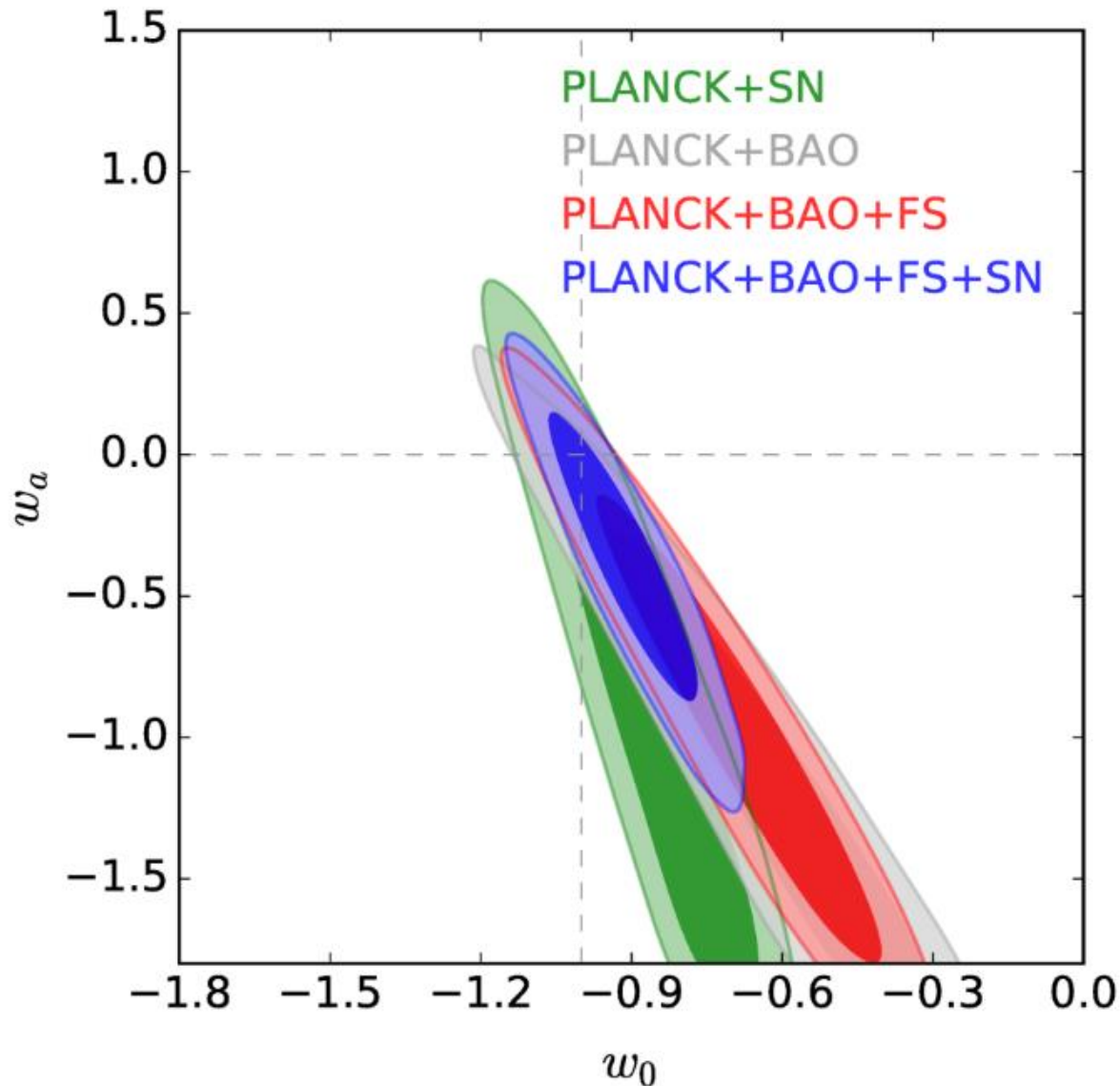


Could be used to address the cosmological constant problem using ghostly copies of the standard (model) fields ? [Linde 1984-1988, Kaplan-Sundrum 2006](#).



To obtain bouncing cosmologies ?
[Brandenberger, Peter, 2017](#).

↪ In cosmology: room for (dominant or weak) energy condition violations with dark energy (e.g. « Phantom DE » [Caldwell, 2002](#))



$$w_{DE} = w_0 + (1 - a) w_a$$

$$= w_0 + \frac{z}{1+z} w_a$$

Constraints on w_0 and w_a from various observations including SNIa (SN) and CMB (PLANCK)

Figure taken from the PDG review on dark energy:

Or from DESI 2024

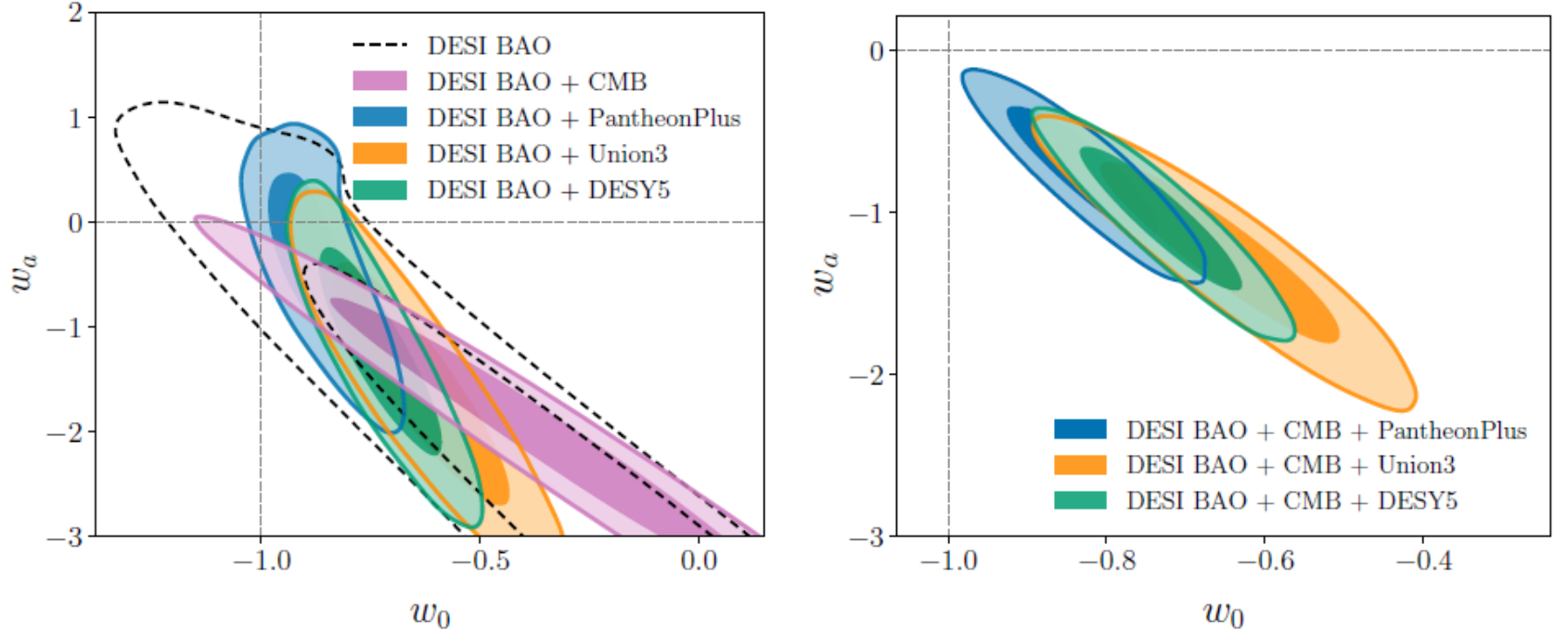


Figure 6. *Left panel:* 68% and 95% marginalized posterior constraints in the w_0 - w_a plane for the flat $w_0 w_a$ CDM model, from DESI BAO alone (black dashed), DESI + CMB (pink), and DESI + SN Ia, for the PantheonPlus [24], Union3 [25] and DESY5 [26] SNIa datasets in blue, orange and green respectively. Each of these combinations favours $w_0 > -1$, $w_a < 0$, with several of them exhibiting mild discrepancies with Λ CDM at the $\gtrsim 2\sigma$ level. However, the full constraining power is not realised without combining all three probes. *Right panel:* the 68% and 95% marginalized posterior constraints from DESI BAO combined with CMB and each of the PantheonPlus, Union3 and DESY5 SN Ia datasets. The significance of the tension with Λ CDM ($w_0 = -1$, $w_a = 0$) estimated from the $\Delta\chi^2_{\text{MAP}}$ values is 2.5σ , 3.5σ and 3.9σ for these three cases respectively.



Some authors have argued that ghosts can be benign, even with runaways

Smilga 2004, 2021; Damour, Smilga, 2021



Some scarce numerical studies indicates that a ghost can stably (global stability) interact with a positive energy d.o.f. or have « islands » of stable initial conditions (also expected from [Kolmogorov-Arnold-Moser](#) (KAM) theorem).

Pagani, Tecchiolli, Zerbini, 1987; Smilga 2005; Carrol, Hoffman, Trodden, 2003; Pavsic, 2016, 2013; Boulanger, Buisseret, Dierick, White, 2019.

II. Ghostifying (integrable) models:

building (integrable) theories of a ghost interacting
with a positive energy degree of freedom



Start with a system with N positive energy degrees of freedom:

N canonical pairs $\xi_i = (x_i, p_i)$

And Hamiltonian

$$H(x_i, p_i) = \sum_{i=1}^N \frac{p_i^2}{2m_i} + V(x_i, \dots x_N)$$



Then consider the complex transformation

$$\mathfrak{C}_{x^n}^{\pm} : \left\{ \begin{array}{ll} x_j \rightarrow \pm i x_j & \text{for } j = n \\ p_j \rightarrow \mp i p_j & \text{for } j = n \\ x_j \rightarrow x_j & \forall j \neq n \\ p_j \rightarrow p_j & \forall j \neq n \end{array} \right\}$$



It preserves the Poisson brackets $\{\}_{\text{PB}}$

$$\begin{aligned} \{\mathfrak{C}_{x^n}^{\pm}(x_i), \mathfrak{C}_{x^n}^{\pm}(p_j)\}_{\text{PB}} &= -\{\mathfrak{C}_{x^n}^{\pm}(p_i), \mathfrak{C}_{x^n}^{\pm}(x_j)\}_{\text{PB}} = \delta_{ij} , \\ \{\mathfrak{C}_{x^n}^{\pm}(x_i), \mathfrak{C}_{x^n}^{\pm}(x_j)\}_{\text{PB}} &= 0 , \\ \{\mathfrak{C}_{x^n}^{\pm}(p_i), \mathfrak{C}_{x^n}^{\pm}(p_j)\}_{\text{PB}} &= 0 . \end{aligned}$$



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But flips the sign of the kinetic energy of the n^{th} d.o.f.

$$p_n \rightarrow \pm i p_n \Rightarrow p_n^2/m_n \rightarrow -p_n^2/m_n$$

Transforming a positive energy d.o.f. (**P**) into a ghost (**G**)

E.g. consider a **two** degree of freedom system with positive kinetic energies and Hamiltonian

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x, y)$$

a « **PP** » system



Then, do the transformation

$$\mathfrak{C}_y^{\pm} \left\{ \begin{array}{l} y \rightarrow \pm i y \\ p_y \rightarrow \mp i p_y \end{array} \right.$$

This transforms the system to a « **PG** » one

$$H = \frac{1}{2}p_x^2 - \frac{1}{2}p_y^2 + V(x, \pm i y)$$



The interaction $V(x, y)$



Under the transformation $\mathcal{C}_y^\pm \left\{ \begin{array}{l} y \rightarrow \pm i y \\ p_y \rightarrow \mp i p_y \end{array} \right.$



$$V(x, \pm i y)$$

Can become complex

But not always true: e.g. $x^2 y^2 \rightarrow -x^2 y^2$ etc....

Now consider an integrable **PP** system

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x, y)$$

I.e. such that there exists an extra integral of motion I (besides the Hamiltonian) :

The value of I is preserved under the motion, i.e.

$$\frac{dI}{dt} = \{I, H\}_{\text{PB}} = 0$$

The preservation of the Poisson brackets,
implies that an such an integrable **PP** system

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x, y)$$

With the extra constant of motion I $\left[\frac{dI}{dt} = \{I, H\}_{\text{PB}} = 0 \right]$



Is transformed under

$$\mathfrak{G}_y^{\pm} \left\{ \begin{array}{l} y \rightarrow \pm i y \\ p_y \rightarrow \mp i p_y \end{array} \right.$$



To an integrable « **PG** » system (with the « same » I)



This can be used to build integrable stable **PG** systems out of integrable **PP** systems

Integrable (mostly **PP**) systems with 2 d.o.f. have been studied and classified since long ago ...

... with the Pioneer work of [Liouville 1855](#), [Darboux 1901](#)

With a Hamiltonian of the form

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x, y)$$



A (full ?) classification based on the **degree** of the integral motion I in the momenta exists up to ... a not very large degree (2?)

([Darboux 1901](#); [Whittaker 1964](#); [Fris, Smorodinskii, Uhler, Winternitz 1967](#); [Holt 1982](#), [Ankiewicz, Pask, 1983](#); [Dorizzi, Grammaticos, Ramani, 1983](#); [Thompson, 1984](#); [Sen, 1985](#); [Hietarinta, 1987](#); [Nagakawa, Yoshida, 2001](#); [Mitsopoulos, Tsampanlis, Paliathanasis, 2020](#)).

At **quadratic** order (in the momenta), we have among others

(as discovered by [Liouville. 1855](#))

$$\left\{ \begin{array}{l} V = \frac{f(u) - g(v)}{u^2 - v^2} , \\ I = - (xp_y - yp_x)^2 - c p_x^2 \\ \quad + 2 \frac{u^2 g(v) - v^2 f(u)}{u^2 - v^2} , \end{array} \right.$$

$$\text{With } \left\{ \begin{array}{l} r^2 = x^2 + y^2 \\ u^2 = \frac{1}{2} \left(r^2 + c + \sqrt{(r^2 + c)^2 - 4 c x^2} \right) \\ v^2 = \frac{1}{2} \left(r^2 + c - \sqrt{(r^2 + c)^2 - 4 c x^2} \right) \\ f \text{ and } g \text{ arbitrary functions} \end{array} \right.$$

III. Stable motion of a ghost interacting
with a positive energy degree of freedom

Consider this « Liouville » integrable model

$$H_{\text{LV}} = \frac{p_x^2}{2} + \sigma \frac{p_y^2}{2} + V_{\text{LV}}(x, y)$$

With

$$\left[\begin{array}{l} V_{\text{LV}} = \frac{f(u) - g(v)}{u^2 - v^2} , \\ u^2 = \frac{1}{2} \left(r^2 + c + \sqrt{(r^2 + c)^2 - 4 c x^2} \right) , \\ v^2 = \frac{1}{2} \left(r^2 + c - \sqrt{(r^2 + c)^2 - 4 c x^2} \right) , \\ r^2 = x^2 + \sigma y^2 . \\ f, g \text{ arbitrary functions, } c \text{ an arbitrary constant} \end{array} \right.$$



The original model is PP (*i.e.* $\sigma = +1$), but by using a proper complex canonical transformation, it can be transformed to a PG one (*i.e.* $\sigma = -1$).

The system has two constant of motion :

The Hamiltonian $\left\{ \begin{array}{l} H_{LV} = \frac{p_x^2}{2} + \sigma \frac{p_y^2}{2} + V_{LV}(x, y) \\ V_{LV} = \frac{f(u) - g(v)}{u^2 - v^2} \end{array} \right.$

And $\left\{ \begin{array}{l} I_{LV} = -\sigma (p_y x - \sigma p_x y)^2 - c p_x^2 + \mathcal{V} \\ \mathcal{V} = 2 \frac{u^2 g(v) - v^2 f(u)}{u^2 - v^2} \end{array} \right.$

Now consider the PG case (with $\sigma = -1$)
and negative c (or positive $\tilde{c} = -c$)



**This makes the momentum-
dependent part of I_{LV} positive**

We prove (for the PG case with negative c)
that the phase-space motion is bounded if

(i) $f(u)$ and $g(v)$ are bounded below, i.e.,

$$\begin{aligned} f(u) &\geq f_0 \\ g(v) &\geq g_0 \end{aligned}$$

with constants $f_0, g_0 \in \mathbb{R}$; and

(ii) at large $|u|$ and $|v|$, these lower bounds sharpen to

$$\begin{aligned} f(u) &\geq 4F_0 |u|^\zeta > 0 \\ g(v) &\geq 4G_0 |v|^\eta > 0 \end{aligned}$$

with positive constants $F_0, G_0 \in \mathbb{R}^+$ as well as $\zeta > 2$
and $\eta > 2$

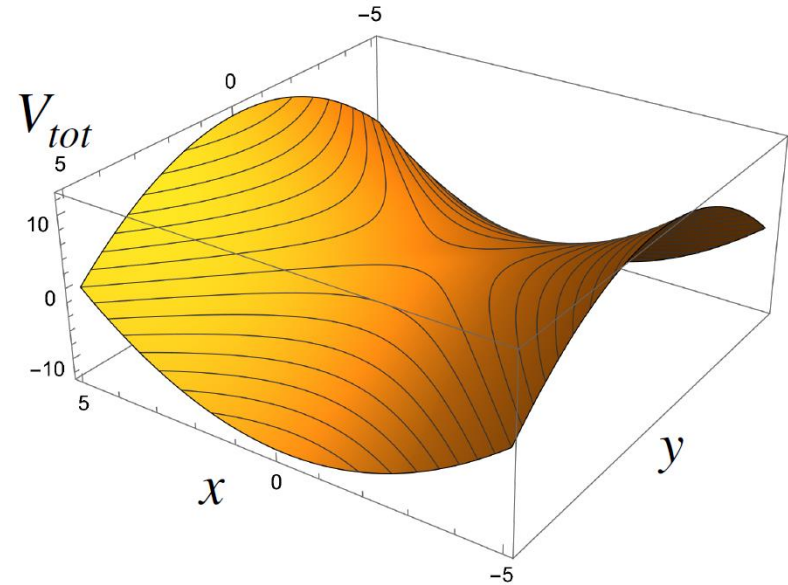
One example is the theory considered in

C.D., . Mukohyama, A. Vikman, PRL 128 (2022) 4, 041301 :

$$H = \frac{1}{2} (p_x^2 + x^2) - \frac{1}{2} (p_y^2 + y^2) + V_I(x, y)$$

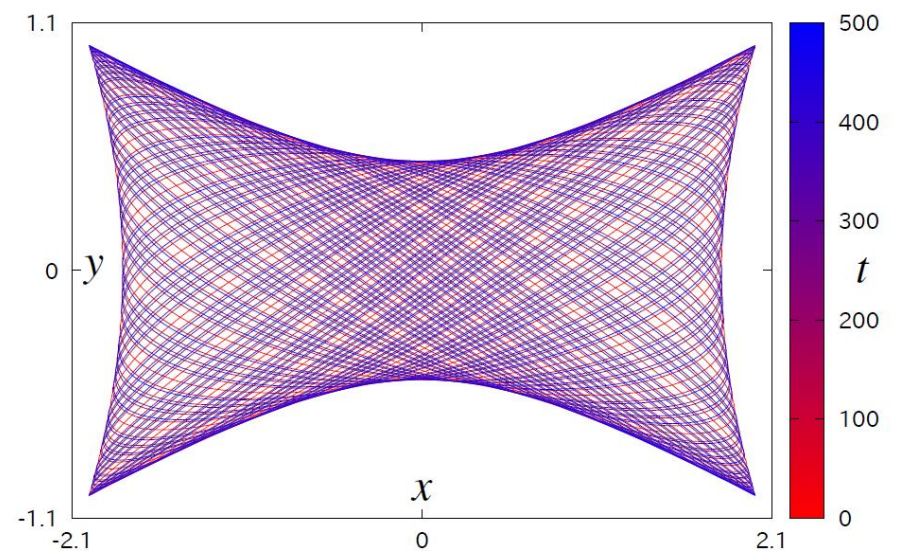
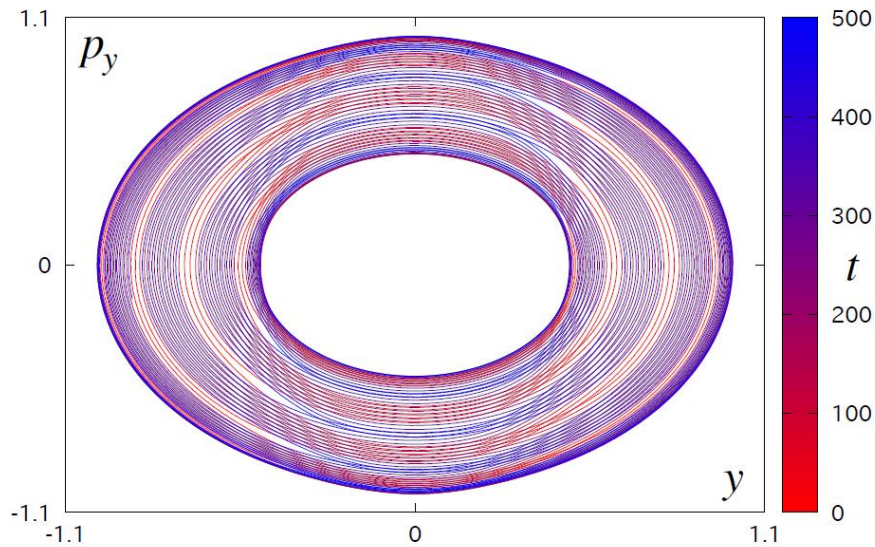
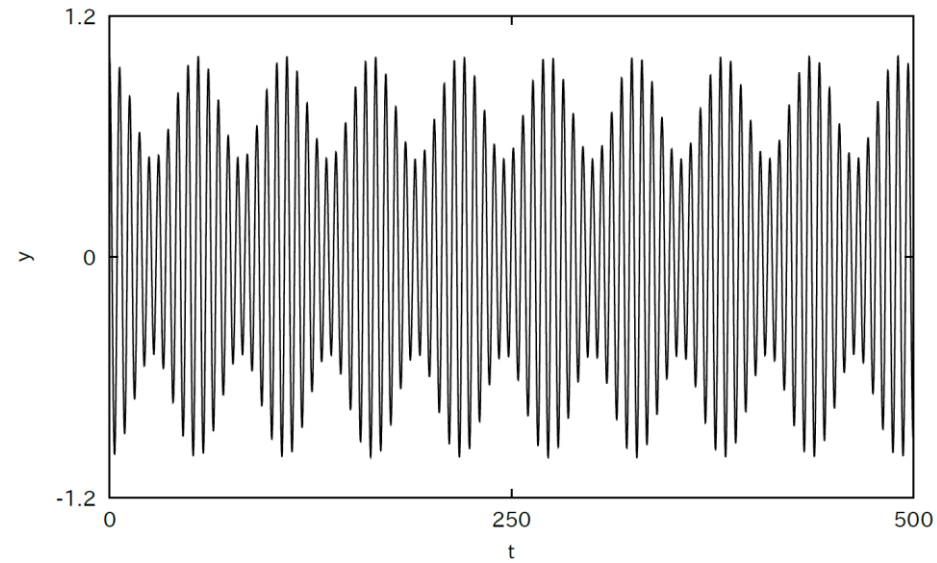
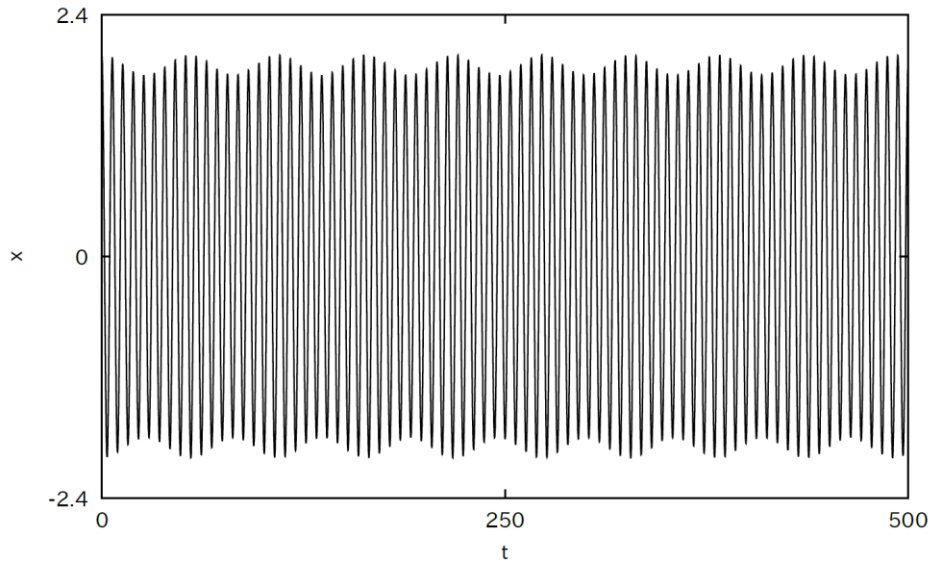
$$\text{With } V_I(x, y) = \lambda \left((x^2 - y^2 - 1)^2 + 4x^2 \right)^{-1/2}$$

and λ a constant



Total potential energy plotted here for
the coupling constant $\lambda = \frac{1}{3}$

This yields indeed a totally stable motion in phase space despite the ghost and the unbounded above and below interaction potential



Can one remove the interactions between the oscillators by making a suitable canonical transformation?

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No !

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No !

This can be shown using a theorem by Arnold:

$$\text{define } \begin{cases} z_x = p_x + ix \\ \bar{z}_x = p_x - ix \end{cases} \quad \dots \text{ and similarly for } y$$

then (in the « non resonant » case, where, as generically true,

$\omega_x/\omega_y = \sqrt{(1-2\lambda)/(1+2\lambda)}$ is irrational) any interaction

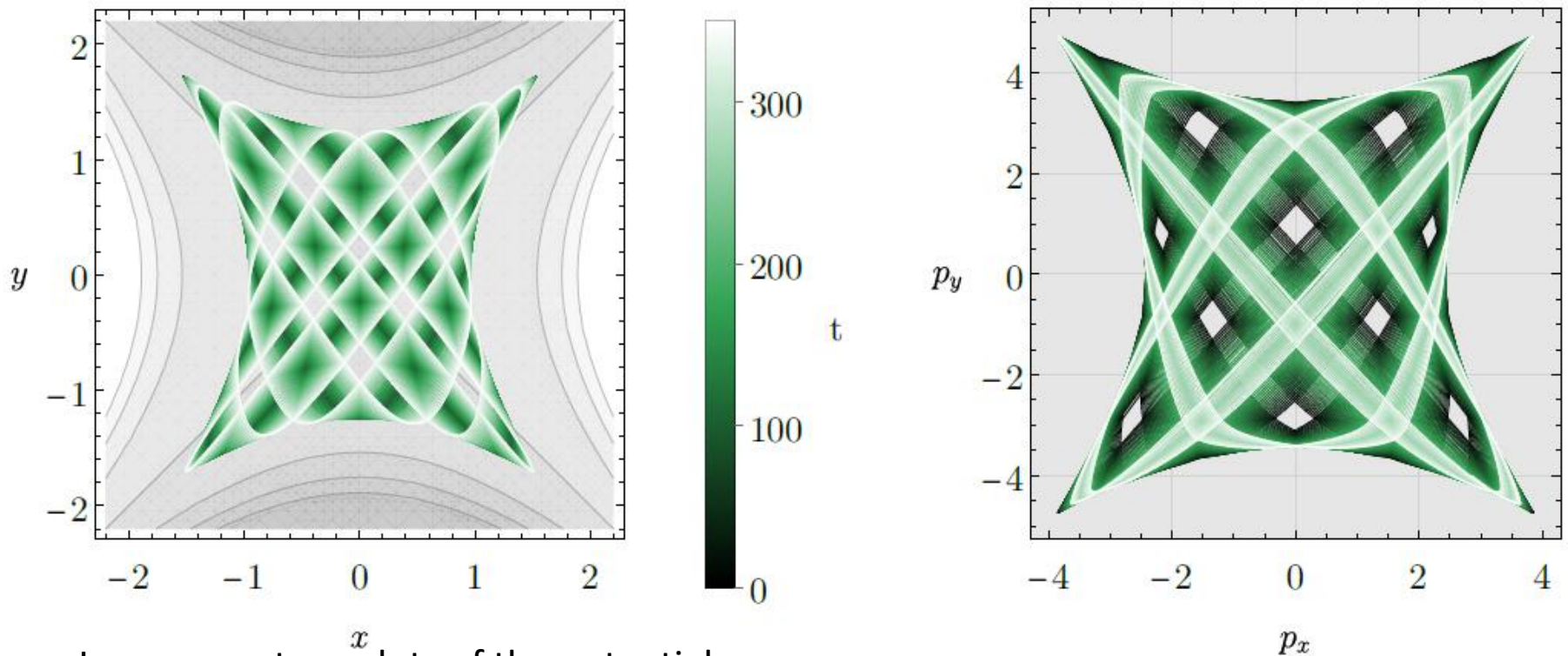
of the form $z_x^{\alpha_x} \bar{z}_x^{\beta_x} z_y^{\alpha_y} \bar{z}_y^{\beta_y}$ can be removed by a canonical

transformation, except if $\alpha_x = \beta_x$ and simultaneously $\alpha_y = \beta_y$

Models with nicer potential can also be found such as

$$V_{\text{LV}}^{(4)}(x, y) = \frac{\omega_x^2}{2} \left[x^2 - \frac{(x^2 - y^2)^2}{c} \right] - \frac{\omega_y^2}{2} \left[y^2 - \frac{(x^2 - y^2)^2}{c} \right] + \mathcal{C}_4 \left[(x^2 - y^2)^3 - c(x^4 - y^4) \right] .$$

A motion for $N = 4$ ($\omega_x^2 = 1$, $\omega_y^2 = 1$, $\mathcal{C}_4 = 1$, and $\tilde{c} = 1$)



In gray: contour plots of the potential

IV. Conclusions

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- Field theory ?

Thank you for your attention !